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CHAOTIC MODES OF OSCILLATION OF A VIBRO-IMPACT SYSTEM†

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The motion of a point mass on a spring with friction and with the condition of absolutely elastic impact against the arresting devices is investigated. The sufficient conditions for chaotic oscillations are derived analytically for the problem considered. The mechanism by which such oscillations arise is described.

The existence of chaotic modes for different vibro-impact systems has recently been established [1, Section 2.4; 2–13]. but in the majority of cases this has been proved numerically. An exception to this is the investigation of the dynamics of a ball, bouncing on a vibrating table [1, Section 2.4]. The mathematical model of this problem is a discrete dynamical system in which the instant of time corresponding to the impact and the rate of impact are chosen as the state variables. The mapping which specifies the dynamical system places in correspondence to this pair of variables a pair corresponding to the next impact in time. It has been shown that for the system considered there is a so-called "Smale horseshoe" [14–16], which ensures chaotic behaviour of the solutions. However, the application of this approach to the problem considered involves considerable difficulties.

Below we describe a new method for the analytical investigation of the behaviour of the solutions of systems with an impact. We choose the classical variables, namely, the coordinate of the point and its velocity, as the phase variables. We will show that solutions corresponding as close as desired to the initial data can have a different number of impacts in a time interval equal to the period of the system considered. It is shown that this fact implies the presence of a Smale horseshoe.

1. FORMULATION OF THE PROBLEM

The general properties of the system. Consider the motion of a point mass along a straight line under the action of a linear recovery force, a linear resistance of the medium and a piecewise-constant periodic stimulating force. We will assume that the point considered impacts absolutely elastically against an arresting device (a stop). The motion of such a point is described by the equation

$$\ddot{x} + 2\varepsilon\dot{x} + x = f(t); \quad f(t) = \begin{cases} 1, & \text{if} \quad t \in [0, T_1) \\ -1, & \text{if} \quad t \in [T_1, T) \end{cases} \tag{1.1}$$

Suppose $f(t)$ is a periodic function of period $T = T_1 + T_2$, defined in the half-interval $[0, T)$ with the above form. Equation (1.1) is specified when $x \geq 0$, and the condition for an absolutely elastic impact is expressed as follows: if $x(t_0) = 0$ while $\dot{x}(t_0 - 0) \leq 0$, then $\dot{x}(t_0 + 0) = -\dot{x}(t_0 - 0)$; if $x(t_0) = 0$, $\dot{x}(t_0 - 0) = 0$ and $kT + T_1 \leq t_0 < (k + 1)T$, then $x(t) = \dot{x}(t) = 0$ when $t_0 \leq t \leq (k + 1)T$. Henceforth we will assume that $\varepsilon \in (0, 1)$, $T_2 > 3T_1$, and the quantity T_1 is a large parameter. In addition to Eq. (1.1) we will consider the equivalent system

$$\dot{x} = y, \quad \dot{y} = -2\varepsilon y - x + f(t) \tag{1.2}$$

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We will assume that system (1.2) is specified in the region

$$\Lambda = \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(0, y) : y \geq 0\}$$

This system or Eq. (1.1) with the indicated impact conditions will be called system A. As follows from results obtained earlier [17], for any t_0, x_0, and y_0, satisfying the conditions $(x_0, y_0) \in \Lambda$, the solution of system A with the initial conditions $x(t_0) = x_0, y(t_0) = y_0$, is defined for all t, uniquely and continuously by the initial data for all values of t such that $x(t) \neq 0$.

We will show that the system considered is dissipative. Consider the positive-definite quadratic form

$$W(x, y) = x^2 + y^2 + 2\varepsilon xy$$

For any solution $(x(t), y(t))$ of system A the function $w(t) = W(x(t), y(t))$ is continuous, including at the impact points. We will consider its derivative by virtue of the system

$$\dot{W} = -2\varepsilon x^2 - 2\varepsilon y^2 - 4\varepsilon^2 xy + 2yf(t) + 2\varepsilon xf(t) = -2\varepsilon W + 2f(t)(y + \varepsilon x)$$

Hence it follows that an $R > 0$ exists such that $\dot{W} < 0$ if $W > R$. Then, all the solutions of system A fall inside the compactum

$$\Xi_R = \{(x, y) \in \mathbb{R}^2 : W(x, y) \leq 2R\}$$

and remain there over time.

The behaviour of the solutions of the system in finite time intervals is determined by the sign of its right-hand side. Consider the equations

$$\ddot{x} + 2\varepsilon\dot{x} + x = 1 \tag{1.3}$$

$$\ddot{x} + 2\varepsilon\dot{x} + x = -1 \tag{1.4}$$

with the same impact conditions. The vibro-impact systems obtained will be called system B and system C respectively. We will conditionally further assume that all the numerical quantities denoted by the letters c and C, are positive constants. We will call the functions $a(t)$ and $b(t)$, defined on the ray, $[t_0, +\infty)$ equivalent $(a(t) \sim b(t))$, if a $t_1 \geq t_0$ exists such that $c|a(t)| \leq |b(t)| \leq C|a(t)|$ for any $t \geq t_1$. We will also use the following standard notation: $a(t) = o(b(t))$, if $a(t)/b(t) \to 0$, and $a(t) = O(b(t))$, if $|a(t)| \leq C|b(t)|$.

In Sections 2 and 3 we will obtain auxiliary results regarding the behaviour of the solutions of system B and C respectively; in Section 4 we will investigate how the solutions of one system transfer into the solutions of another at instants of time when the right-hand side changes sign; in section 5 we will construct a set containing all the non-stray points of the Poincaré mapping of system A and, finally, in the last section we will investigate the structure of the set of non-stray points and we will show that this set contains a Smale horseshoe.

2. THE BEHAVIOUR OF SOLUTIONS IN SECTIONS IN WHICH $f(t) = 1$

The solutions of system B in the intervals between impacts had the form

$$\begin{aligned} x(t) &= 1 + (A-1)\exp(-\varepsilon(t-t_0))\cos(\nu(t-t_0)) + \\ &+ B\exp(-\varepsilon(t-t_0))\sin(\nu(t-t_0)) \\ \nu &= \sqrt{1-\varepsilon^2}, \quad A = x(t_0), \quad B = (y(t_0+0) + \varepsilon x(t_0))/\nu \end{aligned} \tag{2.1}$$

It can be seen that the equilibrium position $x \equiv 1$ of system B is asymptotically stable, while all the remaining solutions of the system converge to this equilibrium position as $t \to +\infty$. This is clear for the solutions of Eq. (13), while for system B it follows from the fact that when there is an impact the distance from the point $(0, y_0)$ to the point $O_1 = (1, 0)$ does not change when y_0 is replaced by $-y_0$. Since system B is autonomous, we can speak of its trajectories.

We will denote by Γ those of them which pass through the point O – the origin of coordinates. Suppose $P = (0, y_1)$ is the point of last transversal intersection of Γ with the Oy axis and D_0 is a closed set, bounded

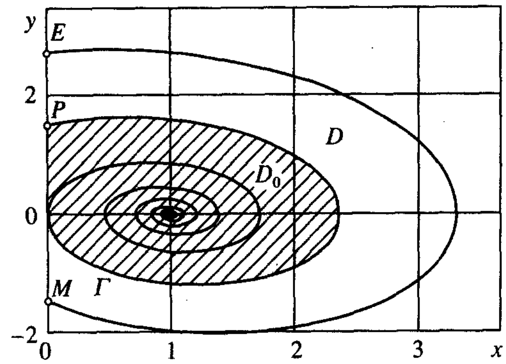

Fig. 1

by the arc OP of the curve Γ and the section of the Oy axis. We extend Γ in the direction of decreasing time to the next intersection with the $x = 0$ axis at the point $E = (0, y_2)$. We take a point $M = (0, -y_1)$ symmetrical to P with respect to the origin of coordinates, and we define D as the closed region bounded by the arc ME of the curve Γ and the corresponding section of the Oy axis. It is clear that $D_0 \subset D$. All the solutions beginning inside the region $\mathrm{Int}D_0$, have no impacts as time increases. The solutions which begin in the region $\mathrm{Int}(D \setminus D_0)$ have exactly one case of impact, after which they fall in the region $\mathrm{Int}D_0$ (see Fig. 1, where the constructions indicated above are given for $\varepsilon = 0.1$).

Suppose $x(t, x_0, y_0)$ is a solution of the Cauchy problem for system B with initial data $(0, x_0, y_0)$. We define the mapping

$$F_1(x_0, y_0) = (x(T_1, x_0, y_0), \dot{x}(T_1 + 0, x_0, y_0))$$

If the value of T_1 is fairly high, the restriction $F_1|_D$ of the mapping F_1 to the set D is continuous and $F_1(D) \subset D_0$. Since the solutions of system B with initial data from D_0 have no impacts as the time increases, for any point $z_{1,2} \in D_0$, the following limit holds

$$\mathrm{dist}(F_1(z_0), F_1(z_1)) \le C\exp(-\varepsilon T_1)\mathrm{dist}(z_0, z_1)$$

3. THE BEHAVIOUR OF THE SOLUTIONS IN SECTIONS IN WHICH $f(t) = -1$

System C also has exactly one fixed point, namely, the origin of coordinates, but the phase portrait of this system differs from the focus. In Fig. 2(a) we show part of the trajectory of system C, corresponding to $\varepsilon = 0.2$ and the initial conditions $x(0) = 0.8$ and $\dot{x}(0) = 0$.

Lemma 1. System C has a unique equilibrium position – the point O, the origin of coordinates. Any solution of system C approaches zero with time.

Proof. Consider the function

$$V = x^2 + 2x + y^2 + 2\varepsilon xy$$

It is positive definite in the Set Λ. Note that for any solution $(x(t), y(t))$ of system C the function $\upsilon(t) = V(x(t), y(t))$ is continuous in t, including at impact points. We will calculate its derivative according to the system considered

$$\dot{\upsilon} = -2\varepsilon y^2 - 2\varepsilon x - 2\varepsilon x^2 - 2\varepsilon^2 xy$$

Hence $-2\varepsilon\upsilon \le \dot{\upsilon} \le -\varepsilon\upsilon$. Hence, $\upsilon(t) = O(\exp(-\varepsilon t))$ and $\exp(-2\varepsilon t) = O(\upsilon(t))$, if $x(t) \equiv 0$.

We define F_2 as the Poincaré mapping for system C during the time T_2. We will trace how the distance between the points changes under the mapping F_2. We fix a certain non-zero solution $z(t)$ of system C, beginning at $t = 0$ in a small neighbourhood of the point $O_1 = (1, 0)$. Note that, by virtue of Lemma 1, the solution $z(t)$ in any finite time interval has a finite number of impacts. Suppose t_k, $k \in N$ are the instants of the impacts and $y_k = y(t_k + 0)$.

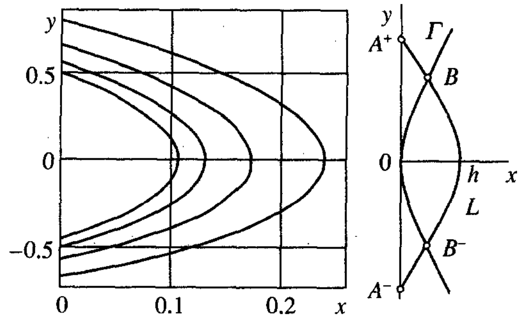

Fig. 2

The following result gives an asymptotic estimate of the rate at which the solution of system C approaches zero with time.

Lemma 2. For any neighbourhood $B \subset D$ of the point O_1 and any $z(0) \in B$

$$y_k/y_1 = \exp(-\varepsilon(t_k - t_1)(2/3 + o(1))) \tag{3.1}$$

where the quantity $o(1)$ approaches zero uniformly with respect to $z(0)$ as $k \to \infty$.

Proof. Consider the function

$$U(x, y) = (x+1)^2 + y^2 + 2\varepsilon y(x+1)$$

The derivative of the function U is equal to $-2\varepsilon U$ by virtue of Eq. (1.4). Fixing the solution $z(t) = (x(t), y(t))$ of system A, we consider the function $u(t) = U(x(t), y(t))$. Then if t_k and t_{k+1} are neighbouring instants of impact, we have

$$u(t_{k+1} - 0) = \sigma_k u(t_k + 0), \quad \sigma_k = \exp(-2\varepsilon(t_{k+1} - t_k)) \tag{3.2}$$

At the same time

$$x(t_k) = x(t_{k+1}) = 0, \quad y(t_k + 0) = y_k, \quad y(t_{k+1} + 0) = -y_{k+1}$$

Hence, Eq. (3.2) can be rewritten in the form

$$y_{k+1}^2 - 2\varepsilon y_{k+1} + 1 = \sigma_k(y_k^2 + 2\varepsilon y_k + 1) \tag{3.3}$$

We will introduce the following notation

$$\delta_k = y_k - y_{k+1}, \quad \Delta_k = t_{k+1} - t_k, \quad X_k = \int_{t_k}^{t_{k+1}} x(t)dt$$

The following estimates hold

$$1 - \sigma_k = 2\varepsilon\Delta_k - 2\varepsilon^2\Delta_k^2 + 4\varepsilon^3\Delta_k^3/3 = O(\Delta_k^4) \tag{3.4}$$

$$y_{k+1} + y_k = \int_{t_k}^{t_{k+1}} (1 + 2\varepsilon\dot{x}(t) + x(t))dt = \Delta_k + X_k \tag{3.5}$$

We will estimate the last term in formula (3.5)

$$X_k = \int_{t_k}^{t_{k+1}} dt \int_{t_k}^{t} ds \left(y_k + \int_{t_k}^{s} (-1 - \varepsilon\dot{x}(\tau) - x(\tau))d\tau \right) = y_k\Delta_k^2/2 - \Delta_k^3/6 + O(\Delta_k^4) \tag{3.6}$$

On the other hand, we can similarly obtain that

$$X_k = y_{k+1}\Delta_k^2/2 - \Delta_k^3/6 + O(\Delta_k^4) \tag{3.7}$$

Taking the half-time of Eqs (3.6) and (3.7) and substituting it into Eq. (3.5), we obtain

$$y_{k+1} + y_k = \Delta_k + \Delta_k^2(y_k + y_{k+1})/4 - \Delta_k^3/6 + O(\Delta_k^4)$$

whence we finally have

$$y_{k+1} + y_k = \Delta_k + \Delta_k^3/12 + O(\Delta_k^4) \tag{3.8}$$

Subtracting Eq. (3.7) from Eq. (3.6), we conclude that

$$\delta_k = O(\Delta_k^2) \tag{3.9}$$

Moreover

$$2y_k = y_k + y_{k+1} + \delta_k = \delta_k + \Delta_k + \Delta_k^3/12 + O(\Delta_k^4) \tag{3.10}$$

It follows from Eq. (3.3) that

$$y_k^2 - y_{k+1}^2 = (1-\sigma_k)(1 + 2\varepsilon y_k + y_k^2) - 2\varepsilon(y_{k+1} + y_k) \tag{3.11}$$

Substituting expressions (3.4), (3.8) and (3.10) into Eq. (3.11) and changing the sign, we obtain, taking relation (3.9) into account,

$$\delta_k\Delta_k = (2\varepsilon\Delta_k - 2\varepsilon^2\Delta_k^2 + 4\varepsilon^3\Delta_k^3/3)(1 + \varepsilon(\delta_k + \Delta_k) + \Delta_k^2/4) -$$
$$- 2\varepsilon(\Delta_k + \Delta_k^3/12) + O(\Delta_k^4) = 2\varepsilon^2\Delta_k\delta_k - 2\varepsilon^3\Delta_k^3/3 + \varepsilon\Delta_k^3/3 + O(\Delta_k^4)$$

Transferring $2\varepsilon^2\Delta_k\delta_k$ to the left-hand side and cancelling against $\Delta_k(1 - 2\varepsilon^2)$, we obtain that

$$\delta_k = \varepsilon\Delta_k^2/3 + O(\Delta_k^3) \tag{3.12}$$

if $\varepsilon \neq 1/\sqrt{2}$. For $\varepsilon = 1/\sqrt{2}$ relation (3.12) is obtained by taking the limit. Note that $y_k = \Delta_k/2 + O(\Delta_k^2)$.
The equivalent form of Eq. (3.12)

$$y_{k+1} = y_k - \delta_k = y_k(1 - 2\varepsilon\Delta_k/3 + O(\Delta_k^2))$$

can be written in the form

$$\ln y_{k+1} - \ln y_k = -2\varepsilon\Delta_k/3 + \xi_k, \quad \xi_k = O(\Delta_k^2) \tag{3.13}$$

Summing Eqs (3.13) we obtain

$$\ln y_n - \ln y_1 = -2\varepsilon(t_n - t_1)/3 + \xi_1 + \xi_2 + \ldots + \xi_{n-1}$$

For arbitrary $\sigma > 0$ we choose a constant $C(\sigma)$ such that the following limit is satisfied

$$\xi_1 + \xi_2 + \ldots + \xi_{n-1} \le C(\sigma) + \sigma(\Delta_1 + \Delta_2 + \ldots + \Delta_{n-1}) = C(\sigma) + \sigma(t_n - t_1)$$

Then

$$\left|\frac{\ln y_n - \ln y_1}{t_n - t_1} - \frac{2}{3}\varepsilon\right| \le \frac{C(\sigma)}{t_n - t_1} + \sigma$$

This also indicates that Eq. (3.1) holds.

We will investigate how the curve Γ divides parts of the trajectories of system C into intervals between impacts.

Suppose L is an arc of the trajectory of system C passing in the neighbourhood of the origin of coordinates and corresponding to the interval between two successive impacts. We will denote by

$A^{\pm} = (0, y_A^{\pm})$ the points of intersection of the arc L with the Oy axis, such that $y_A^{+} > 0 > y_A^{-}$. As follows from Lemma 1, $y_A^{+} \to 0$ as $T_2 \to +\infty$ and $y_A^{-} = -y_A^{+}(1 + o(1))$. In the neighbourhood of the origin of coordinates the curve Γ is given by the equation $x = \gamma(y)$, while the arc L is given by the equation $x = l(y)$. In this cases γ and l are in fact C^1-smooth functions, defined in the section $[y_A^{-}, y_A^{+}]$ and satisfying the following conditions

$$\gamma(0) = l(y_A^{-}) = l(y_A^{+}) = \gamma'(0) = l'(0) = 0, \quad \gamma(y_A^{\pm}) > 0, \quad l(0) > 0$$

and finally $\gamma''(y) > 0$, while $l''(y) < 0$ for any $y \in [y_A^{-}, y_A^{+}]$. This means that if the quantity T_2 is sufficiently large, the arc L and the curve Γ intersect at exactly two points $B^{-} = (x_B^{-}, y_B^{-})$ and $B^{+} = (x_B^{+}, y_B^{+})$, where

$$x_B^{\pm} > 0, \quad y_A^{-} < y_B^{-} < 0 < y_B^{+} < y_A^{+}$$

(see Fig. 2b, where we show the mutual position of the curve Γ and the arc L).

Lemma 3. If $X = (h, 0)$ is the point of intersection of the arc L with the Ox axis, then

$$\lim y_A^{+}/y_A^{-} = \lim y_B^{+}/y_B^{-} = -1; \quad \lim y_A^{+}/y_B^{+} = \sqrt{2} \quad \text{as} \quad h \to 0 \tag{3.14}$$

Proof. The solution of system B with initial conditions

$$x(0) = y(0) = 0$$

in the neighbourhood of the origin of coordinates can be represented in the form

$$y = t + O(t^2), \quad x = t^2/2 + O(t^3)$$

Consequently, in the neighbourhood of the origin of coordinates $\gamma(y) = y^2/2 + O(y^3)$. The solution of system C with initial conditions

$$x(0) = h, \quad y(0) = 0$$

can be written as follows:

$$y = -t + O(t^2), \quad x = h - t^2/2 + O(t^3)$$

whence it follows that

$$l(y) = h - y^2/2 + O(y^3)$$

The quantities $y_A^{\pm}$ are the roots of the function L, and consequently

$$y_A^{\pm} = \pm\sqrt{2h} + o(\sqrt{h})$$

while the quantities $y_B^{\pm}$ are the solutions of the equation $l(y) = \gamma(y)$. Hence

$$y_B^{\pm 2}/2 = h - y_B^{\pm 2}/2 + O(y^3)$$

Then $y_B^{\pm} = \pm\sqrt{h}$, which proves that formulae (3.14) hold.

We will investigate how the distance between the solutions of system C vary with time. Suppose $z_i(t) = (x_i(t), y_i(t))$ $(i = 1, 2)$ are the solutions of system C. Suppose L_i are the trajectories of the points $p_i = z_i(t_0)$, $\bar{L}_i$ are the closures of L_i, while L_i^{+} are the corresponding positive half-trajectories. Clearly

$$\operatorname{dist}(z_i(t), \bar{L}_{3-i}) \sim \operatorname{dist}(z_i(t_0), \bar{L}_{3-i})\exp(-\varepsilon(t - t_0)), \quad i = 1, 2 \tag{3.15}$$

i.e. the trajectories of any two points approach each other exponentially as t increases. Suppose $L_1 = L_2 = L$ and ρ is the arc of the trajectory L, connecting the points p_1 ad p_2 (possibly disconnected). We will call the curvilinear integral

$$d^T(p_1, p_2) = \int_{\rho} ds_L \tag{3.16}$$

the distance between the points p_1 and p_2 along the trajectory L, i.e. the length of the arc ρ. If $L_1 \neq L_2$, we will call the quantity

$$d^{\perp}(p_1, p_2) = \min(\mathrm{dist}(p_1, \bar{L}_2), \mathrm{dist}(p_2, \bar{L}_1))$$

the distance between the points p_1 and p_2 in a perpendicular direction. For each of the points p_i we will consider γ_i – the section of the normal at the point p_i to the trajectory L_i. We will determine the distance between the points p_1 and p_2 along the trajectory from the formula

$$d^T(p_1, p_2) = \min(d_1'(p_1, p_2), d_2'(p_1, p_2))$$

$$d_i'(p_1, p_2) = \min_{q_i \in L_i \cap \gamma_{3-i}} d^T(p_i, q_{3-i}), \quad i = 1, 2$$

Note that since the points p_i and q_i lie on one trajectory, the distances $d^T(p_i, q_i)$ are found from formula (3.16).

Lemma 4. For any two solutions $z_1(t)$ and $z_2(t)$ of system C, such that $z_i(t_0) = p_i \in D$ ($i = 1, 2$), we have the relations

$$\begin{Vmatrix} d^{\perp}(z_1(t), z_2(t)) \\ d^T(z_1(t), z_2(t)) \end{Vmatrix} = \begin{Vmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{Vmatrix} \begin{Vmatrix} d^{\perp}(p_1, p_2) \\ d^T(p_1, p_2) \end{Vmatrix}$$

$$a_{ii} = a_{ii}(t, p_1, p_2) \sim \exp(-\varepsilon(t - t_0)), \quad i = 1, 2$$

$$a_{21} = a_{21}(t, p_1, p_2) = \exp(-\varepsilon(t - t_0))(1/3 + o(1))$$

Proof 1. Suppose $L_1 = L_2$. We will assume that $p_2 = z_1(t_0 + \delta, p_1)$ for certain $\delta > 0$. We will denote by t_k^1 and t_k^2 the corresponding successive instants of impacts, $t_k^i = y_i(t_k^1 - 0)$. Then $t_k^2 = t_k^1 + \delta$ for any $k \in \mathbb{N}$. If $t \notin [t_k^1, t_k^2)$ for any $k \in \mathbb{N}$, we have

$$c\exp(-\varepsilon(t - t_0))\mathrm{dist}(p_1, p_2) \leq \mathrm{dist}(z_1(t), z_2(t)) \leq C\exp(-\varepsilon(t - t_0))\mathrm{dist}(p_1, p_2)$$

and if $t \in [t_k^1, t_k^2)$, then

$$\mathrm{dist}(z_1(t), z_2(t)) \sim y_{k1}$$

as long as $\min_{i=1,2}(t_{k+1}^i - t_k^i) > \delta$. Moreover, for any t

$$d^T(z_1(t), z_2(t)) \sim d^T(p_1, p_2)\exp(-\varepsilon(t - t_0)) \tag{3.17}$$

2. Suppose $L_1 \neq L_2$. Let p_i be points on the trajectories L_i, and let $z_i(t) = (x(t), y(t))$ be solutions of system C with initial conditions $z_i(0) = p_i (i = 1, 2)$. We will assume that $d_3^T(p_1, p_2) = 0$. We will say that the half-trajectory L_1^+ is external with respect to L_2^+ if $y_1^1 = y_1(t_1^1) > y(t_2^1) = t_1^2$. Without loss of generality we will assume that the half-trajectory L_1^+ is external. In intervals where both solutions $x_1(t)$ and $x_2(t)$ do not have impact points,

$$\mathrm{dist}(z_1(t), z_2(t)) \sim \mathrm{dist}(z_1(t_0), z_2(t_0))\exp(-\varepsilon(t - t_0))$$

Consider t_j^i and t_{j+1}^i as neighbouring instants of impacts for the solution $x_i(t)$. Suppose

$$y_k^i = y_i(t_k + 0); \quad i = 1, 2, \quad k = j, j+1$$

We put

$$\tau_1 = \min(t_j^1, t_j^2), \quad \tau_2 = \max(t_{j+1}^1, t_{j+1}^2)$$

In the section $[\tau_1, \tau_2]$ the distance $d^T(z_1(t), z_2(t))$ varies by

$$\rho_j = 2d^{\perp}/y_j^1(1 + o(1)) \tag{3.18}$$

where the time intervals between neighbouring impacts and also the lengths of the arcs between impact points are less for the internal trajectory. The solution $z_2(t)$, going along the internal trajectory, into the section $[\tau_1, \tau_2]$ overtakes the solution $z_1(t)$, going along the external trajectory, i.e.

$$y_2(\tau_2) - y_1(\tau_2) > y_2(\tau_1) - y_1(\tau_1)$$

We will estimate how the distance $d^T(z_1(t), z_2(t))$ varies in the section T_2. We fix $\Theta \le T_2$ and consider the time interval $I = [\Theta - 1, \Theta]$. We introduce the function

$$\lambda(t) \equiv \exp(\varepsilon t/3)$$

By virtue of Lemma 2 the rates y_k^1 of impacts of the solution $x_1(t)$ satisfy the relation

$$y_k^1 = \lambda(-2\Theta(1+o(1))) \tag{3.19}$$

which is a consequence of relation (3.1), if we put $t_1 = 0$ in the latter. At the same time, as was shown above, the length of the section of free motion can be represented in the form

$$t_{k+1}^1 - t_k^1 = 2y_k^1(1+o(1))$$

It follows from the last formula and relation (3.19) that the number of impacts corresponding to the solution $x_1(t)$ in the section I can be estimated from the formula

$$N(\Theta) = \lambda(2\Theta(1+o(1)))$$

From this formula, taking expression (3.18) into account, we obtain

$$\sum_{t_j \in [\Theta-1,\Theta]} \rho_j = \lambda(4\Theta(1+o(1)))d^{\perp}(z_1(t_j), z_2(t_j)) = \lambda(\Theta(1+o(1)))d^{\perp}(p_1, p_2)$$

Suppose Θ_0 is the least integer strictly greater than T_2. Then

$$\sum_{t_j \in [0,T_2]} \rho_j \le \sum_{t_j \in [0,\Theta_0]} \rho_j = \sum_{k=1}^{\Theta_0} \lambda(k(1+o(1)))d^{\perp}(p_1, p_2) = \lambda(T_2(1+o(1)))d^{\perp}(p_1, p_2)$$

On the other hand,

$$\sum_{t_j \in [0,T_2]} \rho_j \ge \sum_{t_j \in [0,\Theta_0-1]} \rho_j = \lambda(T_2(1+o(1)))d^{\perp}(p_1, p_2)$$

Consequently

$$\sum_{t_j \in [0,T_2]} \rho_j = \lambda(T_2(1+o(1)))d^{\perp}(p_1, p_2) \tag{3.20}$$

The assertion of Lemma 4 follows from formulae (3.15), (3.17) and (3.20).

4. SWITCHING INSTANTS

At the instant of time $t = T_1$ the right-hand side of system A changes sign from plus to minus. At the same time, as was shown above, $\operatorname{diam} F_1(D_0) \sim \exp(-\varepsilon T_1)$. We will denote by $G_1(x, y)$ the vector field generated by system B, and we will denote by $G_2(x, y)$ the vector field generated by system C. For the latter system the point O_1 will not be a singular point. It can be seen that $G_2(O_1) \,||\, Ox$.

We will put

$$F(z) = F_2(F_1(z)), \quad z \in \Lambda$$

The mapping is a bisection, continuous at all points of the set D, apart from the inverse image of the straight line $x = 0$, and differentiable at all points of continuity, apart from the arc of the curve Γ serving the boundaries of the sets D and D_0. This follows from the fact that any non-zero solution of system C as well as any solution of system B, the initial data of which does not lie on Γ, corresponds to the case

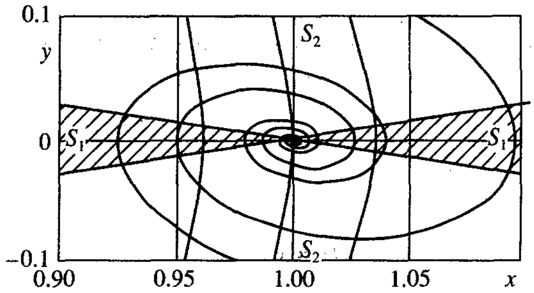

Fig. 3

when the impacts occur only with non-zero velocity. Hence, the locally mapping F can be represented in the form of the composition of a finite number of smooth Poincaré mappings: from the initial point to the first impact, from the first impact to the second, ... , and from the penultimate impact to the last and, finally, from the last impact to the image of the initial point. By virtue of Lemmas 1 and 2 for any $\zeta > 0$ we obtain a number T_0 such that if $T_1 \geq T_0$, then

$$F(D) \subset Q = \{(x, y) : \exp(-(4/3 + \zeta)\varepsilon T_2) \leq V(x, y) \leq \exp(-(4/3 - \zeta)\varepsilon T_2)\} \subset D \qquad (4.1)$$

It can be seen that for sufficiently large T_2 all the non-stray points of the mapping F are points of the set Q. Then,

$$\beta = \max_{(x, y) \in F_1(Q)} \measuredangle\, G_2(x, y), \quad Ox = O(\exp(-\varepsilon T_1)) \qquad (4.2)$$

Here and henceforth $\measuredangle$ is the angle between two smooth directed curves which vary in the section $[0, \pi]$.

For system B, on the other hand, O_1 is a singular point of the focus type, and the directions of the corresponding vector field at points of any neighbourhood of the point O_1 can be arbitrary. Tangents to the trajectories of the system will be parallel to the Ox axis at points of the Oy axis and only at these points. We will fix $\gamma \in (0, \pi/2)$. Consider two sectors

$$S_1 = \{(x, y) : |x - 1| > \operatorname{ctg}\gamma |y|\}, \quad S_2 = \{(x, y) : |x - 1| \leq \operatorname{ctg}\gamma |y|\} \qquad (4.3)$$

(see Fig. 3, where the sectors S_1 and S_2 are constructed for a value of $\gamma = \operatorname{arctg}0.25$, and we also show trajectories of systems B and C corresponding to $\varepsilon = 0.1$).

Quantities $\alpha_1(\gamma)$ and $\alpha_2(\gamma)$ exist such that $\alpha_i(\gamma) = \gamma + o(\gamma)$ and

$$\begin{aligned} &\measuredangle\, G_1(x, y), \quad Ox \leq \alpha_1(\gamma), \quad \text{if} \quad (x, y) \in S_1 \\ &\measuredangle\, G_1(x, y), \quad Ox \geq \alpha_2(\gamma), \quad \text{if} \quad (x, y) \in S_2 \end{aligned} \qquad (4.4)$$

It follows from formulae (4.2) and (4.4) that

$$\measuredangle\, G_1(x, y), G_2(x, y) \leq \alpha_1(\gamma) + \beta, \quad \text{if} \quad (x, y) \in S_1$$

$$\measuredangle\, G_1(x, y), G_2(x, y) \geq \alpha_2(\gamma) - \beta, \quad \text{if} \quad (x, y) \in S_2$$

Suppose p_1 and p_2 are points of the set Q. Suppose γ is the angle between the trajectories of systems B and C at the point p_1, measured in a positive direction, d_1^T is the distance between the points p_1 and p_2 along the trajectories of system B, d_2^T is the distance between the points p_1 and p_2 along the trajectories of system C, and $d_{1,2}^{\perp}$ are the corresponding distances in the perpendicular directions. Then

$$d_2^T = (d_1^T\cos\gamma + d_1^{\perp}\sin\gamma)(1 + o(1)), \quad d_2^{\perp} = |-d_1^T\sin\gamma + d_1^{\perp}\cos\gamma|(1 + o(1))$$

where we mean by $o(1)$ quantities which approach zero as $p_2 \to p_1$.

As regards the behaviour of the trajectories at the instants of time when the right-hand side f of Eq. (1.1) changes sign from minus to plus, it can be seen that

$$\measuredangle\ G_1(x, y), G_2(x, y) - \pi = o(\exp(-2\varepsilon T_2/3 + \zeta\varepsilon T_2))$$

for any point $(x, y) \in Q$.

5. LOCALIZATION OF THE ATTRACTION

In this section we contract the set Q in such a way that the new set, as previously, contains all the non-stray points F. Since, as was shown above, all the solutions of system A, as time passes, fall in the region D and $F(D) \subset Q$, all the non-stray points of the mapping F belong to Q. Note that if T_2 is sufficiently large, then $Q \subset D$ and, consequently, $F(Q) \subset Q$. The sets $Q_k = F^k(Q)$ form a sequence of imbedded compacta $Q \supset Q_1 \supset Q_2 \supset \ldots$. We will put

$$Q_\infty = Q_0 \cap Q_1 \cap Q_2 \ldots$$

The set of non-stray points of the mapping F is then contained in Q_∞.

Suppose the following inequality is satisfied for a certain $\mu > 3$

$$T_2/T_1 > \mu \tag{5.1}$$

We will assume T_1 is so large that, in the definition of the set Q (formula (4.1)), we can put

$$\zeta < \min(1/100, (\mu - 3)/100)$$

We will denote by α the minimum angle between the trajectories of system B and C in the set $F_1(D)$ (note that it depends exclusively on T_1 and ε). By virtue of Lemma 4 the overall length $S(F(Q))$ of the projections of the components of connectedness of the set $F(Q)$ onto the Oy axis is the quantity

$$\sin\alpha\exp(\varepsilon T_2(1/3 + o(1)) - \varepsilon T_1)\mathrm{diam}Q \tag{5.2}$$

We fix a certain number $M > 10$. If T_1 is fairly large, we can estimate a lower bound of the quantity (5.2) by the expression $M\sin\alpha\mathrm{diam}Q$. For M we obtain values of γ_0 and T_1^0 such that if $T_1 > T^0$, $F_1(Q) \subset S_2$ and in the definition of the sector S_2 (formula (4.3)) $\gamma > \gamma_0$, then $M\sin\alpha > 10$. Then

$$S(F(Q)) > 10\mathrm{diam}Q \tag{5.3}$$

and the number of connectedness components of the set $F(Q)$ themselves is less than 10. Note that the greater the value of T_1^0 that is chosen the smaller we can take the quantity γ_0, so that estimate (5.3) remains true.

We will now investigate for what condition $F_1(Q) \subset S_2$. We will introduce ρ and ϕ – polar coordinates in the plane with centre at the point (1.0). It follows from formulae (2.1) that the time interval between successive intersections of the Oy axis by the solution of Eq. (1.3) is equal to π/ν. Moreover, for any such solution $\phi \sim 1$. For points of the set Q the following relation holds

$$\phi - 3\pi/2 = O(\exp(-4(1 + \zeta)\varepsilon T_2/3))$$

The time of the last intersection in the interval $[0, T_1]$ of the sector S_1 by the solutions of system B and the initial data from $(x(0), y(0)) = (x_0, y_0) \in Q$ has an upper limit estimate of $\sigma_0 = C\gamma_0$. It follows from the above that if the quantity T_1 is sufficiently large and in this case

$$T_1 \notin \bigcup_{k=0}^{\infty}\left[\frac{2\pi k}{\sqrt{1-\varepsilon^2}} - \sigma_0, \frac{2\pi k}{\sqrt{1-\varepsilon^2}} + \sigma_0\right] \tag{5.4}$$

condition (5.3) holds (we recall that $\nu = \sqrt{1 - \varepsilon^2}$).

We will estimate the values of the function V in the set $F(Q)$. The diameter Q is estimated to have an upper limit of $2\exp(-2/3 + \zeta/2)\varepsilon T_2)$. Then the diameter $F_1(Q)$ is estimated to be $C_1\exp(-2/3 + \zeta/2)\varepsilon T_2 - \varepsilon T_1)$. We put

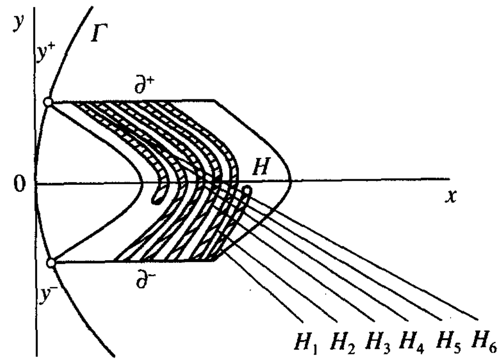

Fig. 4

$$\delta = C_1\exp((-5/3+\zeta)\varepsilon T_2-\varepsilon T_1), \quad \chi = C_1\exp((-1/3+\zeta)\varepsilon T_2-\varepsilon T_1) \tag{5.5}$$

It follows from the above that the constant C_1 can be chosen in such a way that we obtain a point $q \in F_1(Q)$ and a section l_q of the trajectory L_q of system C such that the length l_q is not greater than χ and the whole set $F(Q)$ lies in the δ-neighbourhood of the section l_q. At point of l_q the values of the function V may differ by a factor of no more than $\exp(2\varepsilon\chi)$, which follows from Lemma 1. The value of the function V at each point of the set $F(Q)$ differs from the value at the closest point of the set l_q by no more than 2δ.

Thus, for any two points $p_{1,2} \in F(Q)$ the following limit holds

$$V(p_1) \le V(p_2)\exp(2\varepsilon\chi)+4\delta \tag{5.6}$$

On the other hand, $F(Q) \subset Q$, and for any point of the set Q the corresponding value of V lies in the range

$$I_\upsilon = [\exp(-(4/3+\zeta)\varepsilon T_2), \exp(-(4/3-\zeta)\varepsilon T_2)]$$

Hence, from formula (5.6) we can also obtain that if the value of T_1 is fairly large, we obtain a number $V_0 \in I_\upsilon$ and a $C_2 > 0$ such that

$$V_0 \le V(p) \le V_0 + C_2\exp((-5/3+3\zeta)\varepsilon T_2) = V_1 \tag{5.7}$$

for any point $p \in F(Q)$. We recall that by increasing T_1 we can make the parameter $\zeta > 0$ as small as desired.

6. THE NON-TRIVIAL HYPERBOLIC SET OF NON-STRAY POINTS

If the parameter T_1 is sufficiently large, the Γ curve intersects the curve given by the condition $V(x, y) = V_0$ at two point $Z^+ = (x^+, y^+)$ and $Z^- = (x^-, y^-)$, where $y^- < 0 < y^+$. We will define $H \subset \Lambda$ as the set of points $p = (x, y)$ which satisfy condition (5.7) and such that $y^- \le 0 \le y^+$. Clearly $F(H) \subset Q$, and the mapping $F|H$ is smooth in the neighbourhood of any point of continuity. In this case, in the whole set Q the mapping F is expanding along the trajectories and contracting in a certain transverse direction. Hence, $F|H$ is a local diffeomorfism in the neighbourhood of points of continuity. We will denote by $\partial^\pm$ sections of the boundary of the set H, specified by the conditions

$$y = y^\pm, \quad V_0 \le V(x, y), V_1$$

Suppose $p \in F(H) \cap H$. We will denote by $z_p(t)$ the corresponding solution of system A. We substitute into the corresponding point p the number $n(p)$ of impacts corresponding to the solution z_p in the interval $[-T, 0]$. Suppose N_1 is the minimum value of $n(p)$ in the set $F(H) \cap H$, and N_2 is the maximum value. We put $N = N_2 - N_1 + 1$ and for any $j = 1, \dots, n$ consider the sets

$$H_j = \{p \in F(H) \cap H: n(p) = j + N_1 - 1\}$$

Note that all the sets H_j are compact and that

$$F(H) \cap H = H_1 \cap \ldots \cap H_N$$

(see Fig. 4, where we show the Smale horseshoe for system A for $N = 6$).

Lemma 5. Suppose η is the curve representing the image of the continuous mapping $h : [0, 1] \to H$, such that $h(0) \in \partial^+$, $h(1) \in \partial^-$. Then the set $F(\eta)$ intersects the set H_j for any $j \in \{2, \ldots, N-1\}$, and for any $j \in \{3, \ldots, N-2\}$ the intersection of the sets $F(\eta)$ and H_j contains the curve η_j, which possesses the same properties as η.

Proof. For any point $p \in H$ there is a point $q \in \eta$ such that

$$\operatorname{dist}(p, q) \le C_2 \exp((-5/3 + 3\zeta)\varepsilon T_2)$$

Then the distance from an arbitrary point $F(H)$ to the nearest point of the set $F(\eta)$ can be estimated to have an upper limit of

$$\exp((-4/3 + 4\zeta)\varepsilon T_2 - \varepsilon T_1) = o(\sqrt{V_0})$$

At the same time the quantity $\sqrt{V_0}$ sets a lower limit on the length of the projection of any of the sets H_j onto the Oy axis. It follows that the distance along the trajectories of system C from an arbitrary point of the set $F(H)$ to the nearest point of $F(\eta)$ is not greater than half the length of the projection of H_2. This also indicates that the assertion proved is correct.

We will show that the number N increases without limit as T_1 increases. We will conditionally assume that the curve η^0, given by the conditions $V(x, y) = V_0$ and $y \in [y^-, y^+]$, is directed along decreasing values of the variable y. Suppose l is the length of this curve. As follows from the above, the sum of the lengths of the components of the curve $F(\eta^0)$ is equal to

$$\sin\alpha \exp(\varepsilon T_2(1 + o(1))/3 - \varepsilon T_1)l$$

At each point the direction of the curve $F(\eta^0)$ is close to the direction of the vector field of system C, whence it follows that the length of each component of connectedness of $F(\eta^0)$ is estimated to have an upper limit of Cl. This means that the number N of connectedness components increases without limit. It follows from the assertion of Lemma 3 that $N \ge 10\sqrt{2} > 6$, if limit (5.3) holds.

Lemma 6. If the value of the parameter T_1 is sufficiently large, then for any $m \in N$ and any set of numbers $a = (a_0, \ldots, a_m)$, such that $3 \le a_j \le N-2$ for any $j = 0, \ldots, m$, the set

$$H_a = H_{a_0} \cap F^{-1}(H_{a_1}) \cap \ldots \cap F^{-m}(H_{a_m})$$

is non-empty.

Proof. We fix the subscript a. It follows from Lemma 5 that the image of the curve of η^0 considered above contains the curve $\eta_{a_0} \in H_{a_0}$, connecting ∂^+ and ∂^-. Applying Lemma 5 to the curve of η_{a_0}, we obtain that the curve $\eta_{a_0 a_1} \subset F(\eta_{a_0}) \cap H_{a_1}$ exists. In the final analysis we obtain the curve $\eta_{a_0 \ldots a_m} \subset H(a_m) \cap \ldots \cap F^m(H_{a_0})$. Then the assertion of the lemma follows from the fact that $F^{-m}\eta_{a_0 \ldots a_m} \subset H_a$.

We put

$$K = \bigcap_{n=-\infty}^{\infty} F^n(H)$$

Obviously the set K is invariant under the mapping F, compact and non-empty, like the intersection of the imbedded compacta (the intersections of a finite number of iterations of the compactum H). Moreover, $K \subset F^{-1}(H)$, and consequently the set K is not intersected by the inverse image of the Oy axis. Hence, we obtain a neighbourhood Ω of the set K such that $F|_\Omega$ is a diffeomorfism.

To each point $p \in K$ here corresponds the unique sequence

$$a(p) = \{\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots\}, \quad a_n = \{2, \ldots, N-1\}, \quad n \in \mathbb{Z}$$

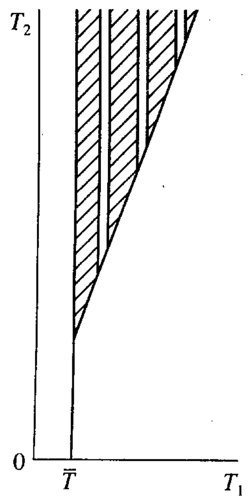

Fig. 5

such that $F^n(p) \in H_{a_n}$ for any $n \in \mathbb{Z}$. It follows from Lemma 6 that for any sequence a one can choose a corresponding point p. Note that, by virtue to the fact that the diffeomorfism F is hyperbolic in the neighbourhood of the set K the point $p \in K$ is uniquely defined by the sequence $a(p)$. If $N \geq 6$, the set of possible values of a_j is not less than 2, and the set K has a power continuum. A shift of the sequence $a(p)$ by unity to the left corresponds to the mapping F. Hence, the mapping $F|K$ possesses the same properties as the famous Smale horseshoe, namely:

(1) the mapping $F|K$ has an infinite number of periodic points;
(2) the periodic points F are always dense in K;
(3) a point $p \in K$ exists, the orbit of which $\{F^n(p) : n \in \mathbb{Z}\}$ is everywhere dense in K.

Thus, the following assertion holds.

Theorem. For any $\varepsilon \in (0, 1)$, $\sigma_0 > 0$ and $\mu > 3$ a $\bar{T} > 0$ exists such that if $T_1 > \bar{T}$, conditions (5.1) and (5.5) hold, and then the mapping F has a hyperbolic invariant set K with properties 1–3.

Hence, we have shown that system A has a compact hyperbolic invariant set containing an infinite number of periodic solutions and an everywhere dense trajectory. Such sets are often called chaotic sets. It can be seen that the set Q_∞ introduced above is an attractor. The chaotic set is, of course, contained in this attractor. These attractors are often called strange attractors. Thus, we have shown that the vibro-impact system A has a strange attractor.

In Fig. 5 the set of values of the parameters T_1 and T_2 corresponding to chaotic oscillations is shown hatched.

We wish to dedicate this paper to the 80th birthday of Academician N. N. Krasovskii.

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